Neural Operators for Excitable Media Dynamics in FitzHugh-Nagumo Systems

Andrew Franck

Overview

- The FitzHugh-Nagumo (FHN) system models dynamics in cardiac and neural tissue via coupled reaction-diffusion PDEs
- While traditional methods require re-simulation for every parameter change, Fourier Neural Operators (FNO) learn solution operators directly
- FNO respect the dynamics of the FHN system, with <5% mean squared error and 900x speedup over traditional numerical solvers
- Maintains relatively stable long-term rollouts, with ~14% mean squared error over 50+ timesteps

Methodology

- Partial Differential Equations (PDEs) describe the evolution of a physical system over time and space by relating partial derivatives to a solution function
- The FitzHugh-Nagumo (FHN) system models dynamics through coupled reaction-diffusion PDEs for activator (u) and inhibitor (v) variables, used in computational neuroscience to study action potential propagation:

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + u - \frac{u^3}{3} - v + I_{\text{ext}}(\boldsymbol{x}),$$
$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + \frac{1}{\tau}(u + a - bv),$$

- Traditional numerical solvers must solve the PDE for each parameter set, requiring hours per simulation. Fourier Neural Operators (FNOs) learn the solution operator, *i.e.,* the mapping from initial conditions and parameters to system evolution.
- In the case of the FHN system, we aim to learn an operator:

$$\mathcal{G}_{\boldsymbol{\lambda}} : \mathcal{U} \times \mathbb{R}^5 \rightarrow \mathcal{V}$$

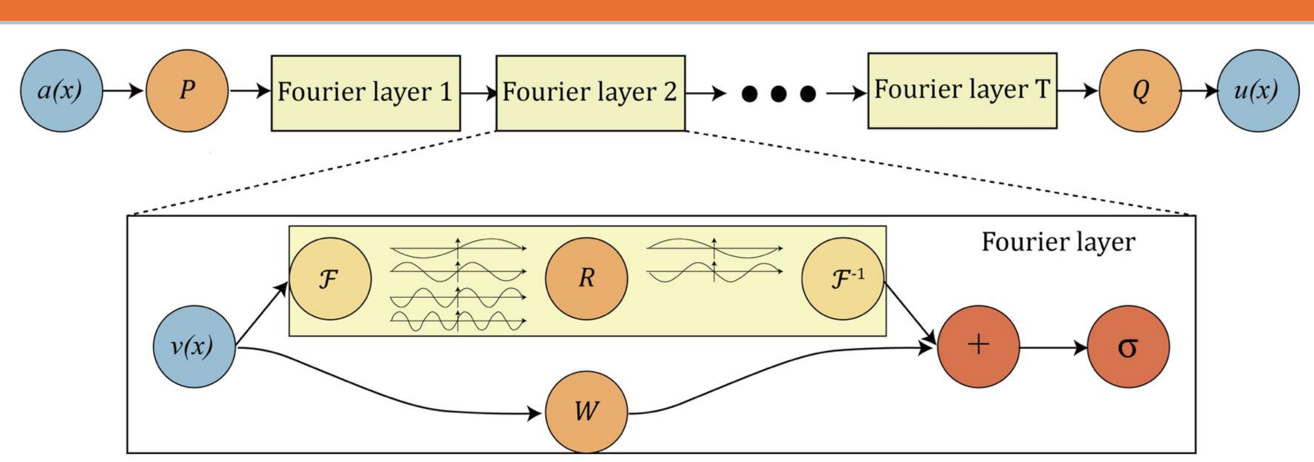
Mapping initial states to future states across the five-parameter FHN system:

$$\boldsymbol{\lambda} = (D_u, D_v, a, b, \tau)$$

where D_u, D_v are the diffusion coefficients, a, b determine reaction kinetics, and τ is the timescale separation between u and v.

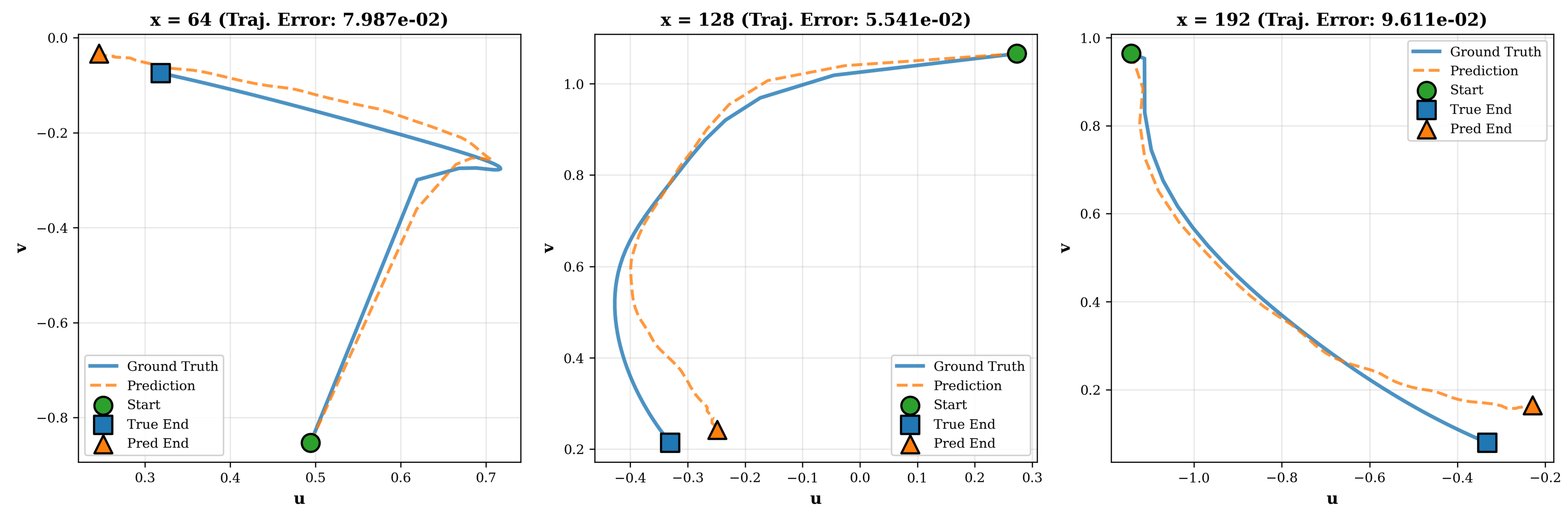
- FNOs take discretized spatial fields (u_t, v_t) and parameters λ as input. By operating in Fourier space on continuous function representations, they are resolution invariant. This enables prediction at any grid resolution, including finer than training data.

FNO Architecture

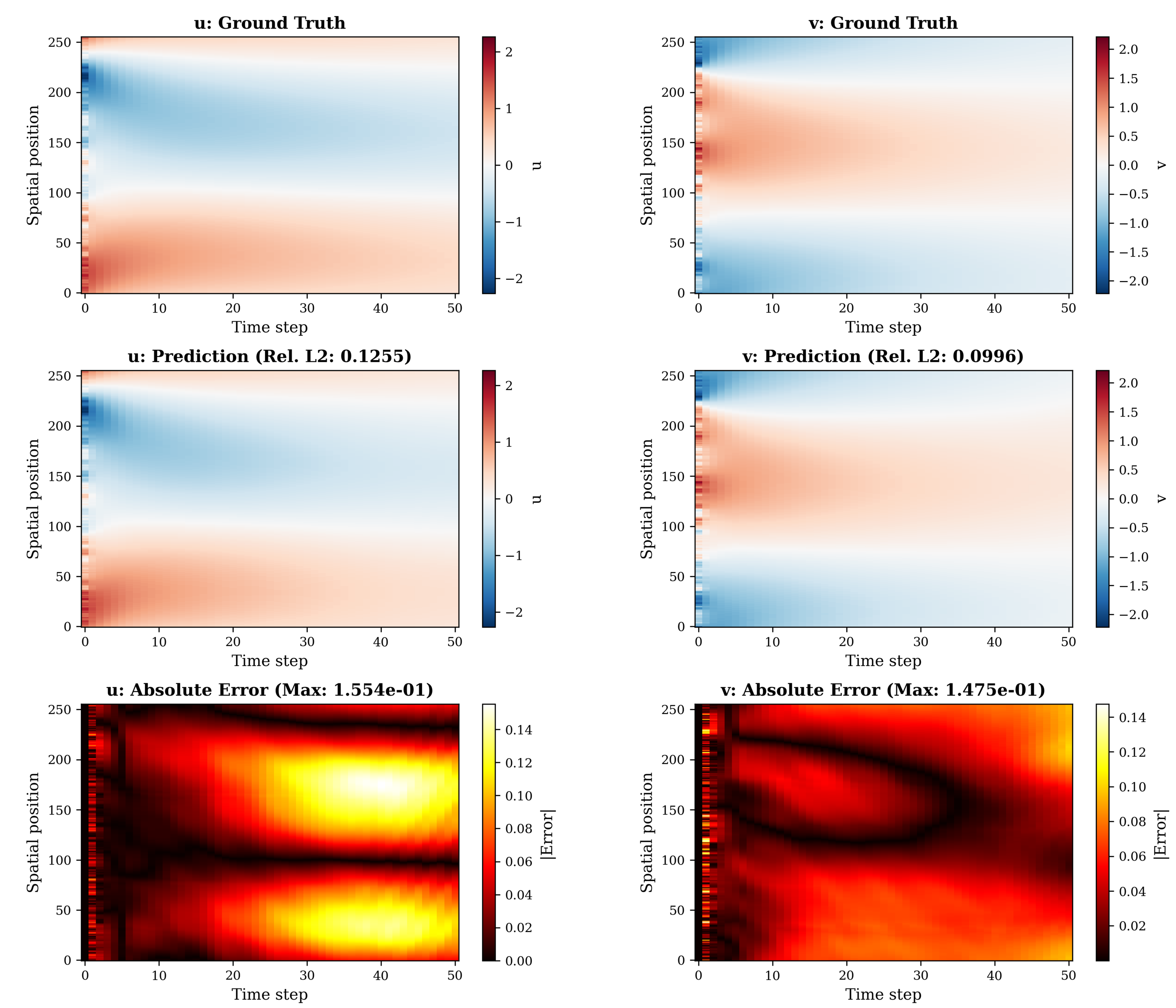


- **Network:** input *a(x)* lifted to higher dimension via fully connected layer *P* to Fourier layers, then projection layer *Q* to output *u(x)*.
- **Fourier Layer:** Dual path combines spectral convolution (FFT, linear transform, inverse FFT) with a local linear transform, then both are summed and activated via. ReLU[1].

Multi-Step Predictions



Phase space portraits at three spatial locations showing *v* vs. *u* over *t* = 50 timesteps. Solid: ground truth, dashed: FNO predictions. Markers show initial condition (green) and endpoints (blue/orange).



Spatiotemporal Heatmap: 50-step rollout comparison. Top row (a-c): ground truth *u*, FNO prediction, absolute error. Bottom row (d-f): corresponding *v* plots. *Axes:* time (horizontal), space (vertical).

Numerical Results

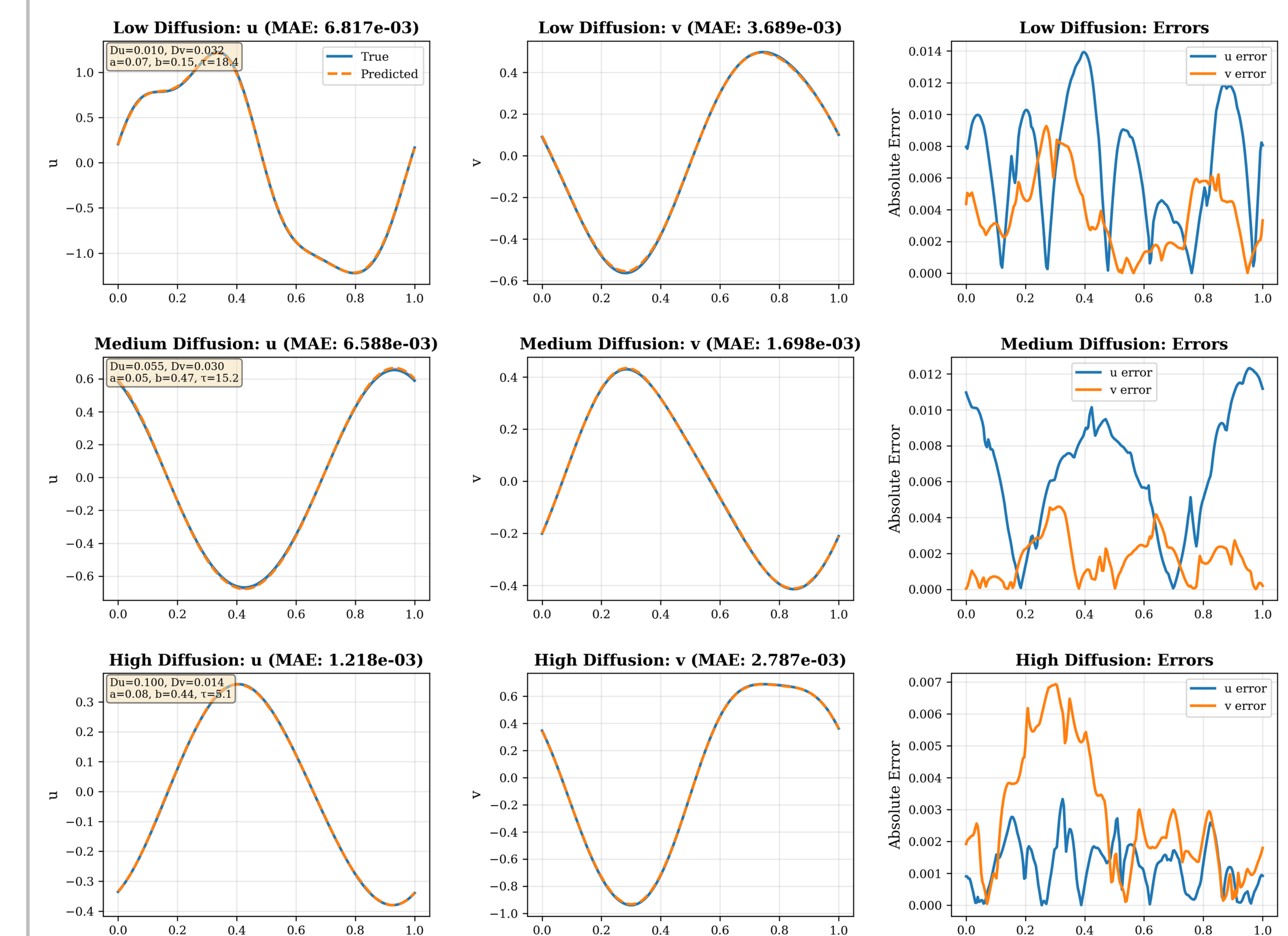
Method	Time per Step	Total Time (50 steps)
Finite-Difference (CPU)	42.3 ± 1.8 ms	2.12 s
FNO (GPU, batch=1)	0.38 ± 0.02 ms	0.019 s
FNO (GPU, batch=32)	0.045 ± 0.003 ms	0.0023 s
Speedup Factor	111× (single)	111×
	940× (batch)	920×

Computational Time: FNO vs. finite-difference solver for 50-step test. Benchmarked on NVIDIA RTX 5080 GPU (FNO) and AMD Ryzen 7 7800X3D CPU (FD solver).

Parameter Regime	$\epsilon_{\text{rel}}(u)$	$\epsilon_{\text{rel}}(v)$
Within range (validation)	0.0098	0.0087
Low diffusion ($D_u = 0.005$, $D_v = 0.002$)	0.0143	0.0126
High diffusion ($D_u = 0.15$, $D_v = 0.08$)	0.0167	0.0139
High time-scale separation ($\tau = 30$)	0.0214	0.0178
Low recovery ($b = 0.05$)	0.0128	0.0114

Parameter extrapolation test: relative L2 error for unseen parameter combinations vs. validation set (mean of 100 trajectories).

Single-Step Predictions



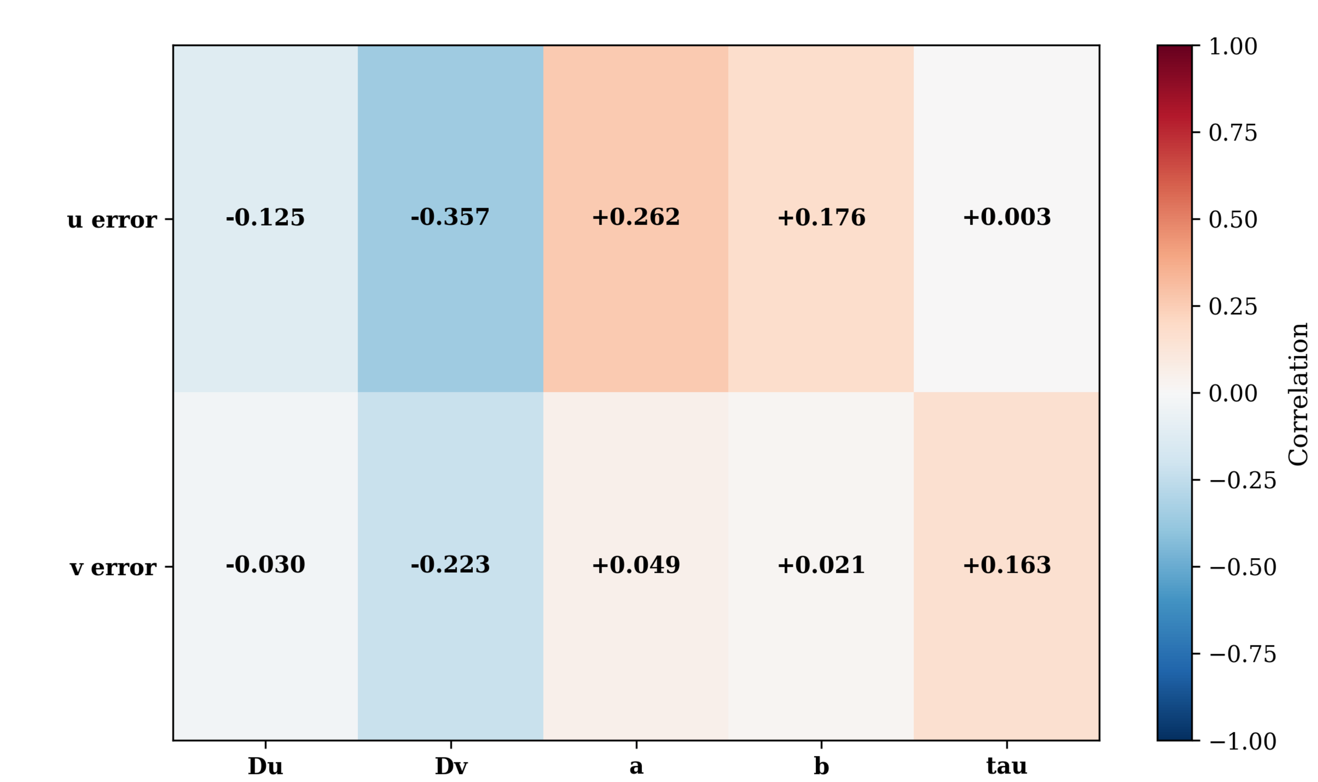
Single Step Prediction Graph: Predictions across diffusion regimes: *u*, *v* predictions and errors for weak, moderate, and strong diffusion parameters.

Single-Step Numerical Results

Metric	Activator (u)	Inhibitor (v)
Relative L^2 Error	0.0754 ± 0.0043	0.0996 ± 0.0039
Mean Squared Error	0.00478 ± 0.00231	0.00212 ± 0.00118
Mean Absolute Error	0.0541 ± 0.0187	0.0392 ± 0.0156
Max Absolute Error	0.1554 ± 0.0421	0.1475 ± 0.0398
R^2 Score	0.9843 ± 0.0089	0.9901 ± 0.0067
Pearson Correlation	0.9948 ± 0.0024	0.9976 ± 0.0016

Validation set performance for single-step predictions (1600 trajectories). Metrics show mean ± standard deviation across all validation samples space.

Parameter Generalization



Generalization: Parameter-error correlation matrix quantifying relationship between FHN parameters and prediction accuracy. Each cell shows the relative L2 error for either the activator *u* or inhibitor *v* (rows). Blue indicates negative correlation, red indicates positive correlation, and white indicates negligible correlation.

1 - Li, Zongyi, et al. "Fourier neural operator for parametric partial differential equations." *arXiv preprint arXiv:2010.08895* (2020).

franck@oxy.edu